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Solubility of additive sextic forms over ramified quadratic extensions of $\mathbb{Q}_2$

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Abstract. In this article, we study the equation $a_1x_1^6 + a_2x_2^6 + \cdots + a_sx_s^6 = 0$ over the six ramified quadratic extensions of the p-adic field $\mathbb{Q}_2$. For all of these extensions, we show that if $s \geq 9$, then this equation has a nontrivial solution regardless of the values of the coefficients. For four of the extensions, we show that 9 is the smallest number of variables that guarantees that the equation will have a nontrivial solution.

1. Introduction

In this article, we are interested in nontrivial $\mathfrak{p}$-adic zeros of additive forms. Specifically, we are interested in nontrivial $\mathfrak{p}$-adic solutions of equations of the form

$$a_1x_1^d + a_2x_2^d + \cdots + a_sx_s^d = 0, \tag{1}$$

where the coefficients lie in a $\mathfrak{p}$-adic field. While studying a conjecture commonly attributed to Artin (see the introduction to [1]), Davenport & Lewis [4] proved that (1) has nontrivial solutions in each of the fields $\mathbb{Q}_p$ provided only that $s \geq d^2 + 1$, and gave examples to show that if $d + 1 = p$ for some prime p then there are additive forms in d^2 variables which do not have nontrivial p-adic zeros.

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However, if $d+1$ is composite, then a smaller number of variables suffices. To express this compactly, write $\Gamma^*(d, K)$ to represent the smallest number of variables which guarantees that (1), with coefficients in the field K, has nontrivial solutions in K regardless of the coefficients. Then the work of Davenport & Lewis shows that $\Gamma^*(d, \mathbb{Q}_p) \leq d^2+1$ for all degrees d and primes p, with equality when $d = p-1$.

These values of $\Gamma^*(d, \mathbb{Q}_p)$ immediately led Davenport & Lewis to the following result [4]. Suppose that the coefficients of (1) are (ordinary) integers, and define $\Gamma^*(d)$ to be the smallest number of variables which guarantees that (1) has a nontrivial solution in every p-adic field $\mathbb{Q}_p$. Then $\Gamma^*(d) \leq d^2 + 1$, with equality whenever $d+1$ is prime. Since [4] was published, several authors (see for example [2], [3], [6], [9], [10]) have studied the exact values of $\Gamma^*(d)$ for various degrees d. Currently, the exact value of $\Gamma^*(d)$ is known for all $d \leq 32$.

Recently the author, inspired by results of Bovey [3], proved the following exact formula [8] for the values of $\Gamma^*(d, \mathbb{Q}_2)$.

Theorem. *Write $d = 2^\tau d_0$, where d_0 is an odd integer, and define the number γ by*

$$\gamma = \gamma(d) = \begin{cases} 1 & \text{if } \tau = 0; \\ \tau + 2 & \text{if } \tau > 0. \end{cases}$$

Further, write $d = \gamma q + r$, where q and r are integers with $0 \leq r \leq \gamma - 1$. Then we have

$$\Gamma_2^*(d) = \begin{cases} 5 & \text{if } d = 2; \\ (2^\gamma - 1)\, q + 2^r & \text{otherwise.} \end{cases}$$

In this article, we take the first steps in an attempt to extend the above theorem to algebraic extensions of $\mathbb{Q}_2$. In particular, we study the values of $\Gamma^*(6, K)$, where K is one of the six ramified quadratic extensions of $\mathbb{Q}_2$. We prove the following theorem.

Theorem 1. *We have*

$$\begin{aligned} \Gamma^*(6, \mathbb{Q}_2(\sqrt{2})) &= 9, & \Gamma^*(6, \mathbb{Q}_2(\sqrt{-2})) &= 9, \\ \Gamma^*(6, \mathbb{Q}_2(\sqrt{10})) &= 9, & \Gamma^*(6, \mathbb{Q}_2(\sqrt{-10})) &= 9, \\ 7 \le \Gamma^*(6, \mathbb{Q}_2(\sqrt{-1})) &\le 9, & 7 \le \Gamma^*(6, \mathbb{Q}_2(\sqrt{-5})) &\le 9. \end{aligned}$$

Note that for our final two fields, $\mathbb{Q}_2(\sqrt{-1})$ and $\mathbb{Q}_2(\sqrt{-5})$, we are able to prove that 9 variables suffice to guarantee solubility, but cannot prove that this is the minimum such value. In fact, our studies suggest that the actual values of $\Gamma^*(6, K)$ are smaller for these fields.

Conjecture 2. *We have*

$$\Gamma^*(6, \mathbb{Q}_2(\sqrt{-1})) = \Gamma^*(6, \mathbb{Q}_2(\sqrt{-5})) = 7.$$

As we will see below, we are able to find specific forms in 6 variables which have no nontrivial zeros in these fields. However, we have been unable to find forms in 7 variables without nontrivial zeros. Some preliminary work indicates to us that it should be possible to prove that no such forms exist.

2. Preliminary concepts

In this section, we describe the main concepts and notation that will be used throughout the proof. We begin by choosing a uniformizer π for each of the fields K under consideration. The table below gives the value of π which we choose for each field, as well as some other information which we will find useful later. Once we have chosen a uniformizer, any integer c of K may be written in the form

$$c = c_0 + c_1\pi + c_2\pi^2 + c_3\pi^3 + \cdots,$$

where we have $c_i \in \{0, 1\}$ for each i.

K	π	$2 \pmod{\pi^5}$	Nonzero 6th powers in $\mathcal{O}_K \pmod{\pi^5}$
$\mathbb{Q}_2(\sqrt{2})$	$\sqrt{2}$	π^2	1 and $1+\pi^2+\pi^3$
$\mathbb{Q}_2(\sqrt{-2})$	$\sqrt{-2}$	$\pi^2+\pi^4$	1 and $1+\pi^2+\pi^3$
$\mathbb{Q}_2(\sqrt{10})$	$\sqrt{10}$	π^2	1 and $1+\pi^2+\pi^3$
$\mathbb{Q}_2(\sqrt{-10})$	$\sqrt{-10}$	$\pi^2+\pi^4$	1 and $1+\pi^2+\pi^3$
$\mathbb{Q}_2(\sqrt{-1})$	$1+\sqrt{-1}$	$\pi^2+\pi^3$	1 and $1+\pi^2+\pi^3+\pi^4$
$\mathbb{Q}_2(\sqrt{-5})$	$1+\sqrt{-5}$	$\pi^2+\pi^3+\pi^4$	1 and $1+\pi^2+\pi^3+\pi^4$

Table 1. Uniformizers and other information for our fields

Because we are considering homogeneous forms, if a nontrivial zero in K exists, then we may "clear denominators" and find a nontrivial zero in the ring of integers $\mathcal{O}_K$. So our goal will always be to find $\mathfrak{p}$-adic integral solutions. We can also obviously assume that the coefficients of our form are in $\mathcal{O}_K$. Suppose that x is one of the variables in our form. We can write the coefficient of x^6 as

$$c = \pi^r(c_0 + c_1\pi + c_2\pi^2 + c_3\pi^3 + \cdots),$$

where r is a nonnegative integer and $c_0, c_1, \ldots \in \{0, 1\}$, with $c_0 = 1$. We will refer to the number r as the *level* of x. We also define the *π-coefficient of x* and the *π-coefficient of c* to both be the number c_1. Thus every variable in F will have a π-coefficient of either 0 or 1.

We now summarize some of the work of Davenport & Lewis in [4], specialized to our situation. Let F be the additive form

$$F = a_1x_1^6 + a_2x_2^6 + \cdots + a_9x_9^6. \tag{2}$$

First, suppose that some variable x in F is at a level $r \geq 6$. Writing $r = 6a + b$ with $0 \leq b \leq 5$, we can write the x-term of F as $c\pi^{6a+b}x^6 = c\pi^b(\pi^a x)^6$, where c is a unit in $\mathcal{O}_K$. Making the change of variables $y = \pi^a x$ yields a new form F' which has nontrivial zeros in $\mathcal{O}_K$ if and only if F does. Moreover, the new variable y is at level at most 5. In this manner, we may assume that every variable of F is at level at most 5.

Our next lemma allows us to assume that our form has certain other desirable properties as well.

Lemma 3. *Let F be an additive form as in* (2) *in which every variable is at level at most 5. Suppose that (possibly after relabeling the variables) we make a nonsingular linear change of variables of the form*

$$F' = \frac{1}{\pi^r} F(\pi x_1, \ldots, \pi x_t, x_{t+1}, \ldots x_9), \tag{3}$$

so that F' also has coefficients in $\mathcal{O}_K$, and that every variable in F' is still at level at most 5. Then the form F' has a nontrivial zero in $\mathcal{O}_K$ if and only if F does. The form F' can be written as

$$F' = F_0' + \pi F_1' + \cdots + \pi^5 F_5',$$

where each F_i' is an additive form in m_i variables, the variables in each form F_i' are distinct, and if x_j is a variable in the form F_i', then the coefficient of x_j in F_i' is not divisible by π. Finally, we may choose the change of variables so that we additionally have the following system of inequalities:

$$\begin{array}{lcl} m_0 & \geq & 2 \\ m_0 + m_1 & \geq & 3 \\ m_0 + m_1 + m_2 & \geq & 5 \\ m_0 + m_1 + m_2 + m_3 & \geq & 6 \\ m_0 + m_1 + m_2 + m_3 + m_4 & \geq & 8 \\ m_0 + m_1 + m_2 + m_3 + m_4 + m_5 & = & 9. \end{array} \tag{4}$$

While Davenport & Lewis only prove these results for the p-adic fields $\mathbb{Q}_p$, their proofs apply to finite extensions without change. If F has coefficients in $\mathcal{O}_K$ and satisfies the system (4), then we will say that F is π-*normalized*.[1] Note that

[1] In [4], Davenport & Lewis do not use the term 'normalized' to refer to a form with these properties, but give a more complicated definition of a normalized system of forms in [5]. Under that definition, a normalized "system" of one form will have the properties mentioned here, although these properties do not guarantee that the form meets their definition of normalized.

in a π-normalized form, every variable is at level at most 5.

Next, we define the concept of a contraction, which is the key to our proof. Suppose that F is an additive form as in (2), and that we have some variables, say $x_1, \ldots, x_t$, which are at (possibly different) levels at most $j-1$. Suppose further that we can find elements $b_1, \ldots, b_t \in \mathcal{O}_K$ such that

$$a_1 b_1^6 + \cdots + a_t b_t^6 = \pi^j m,$$

for some m which is not divisible by π. Then setting $x_i = b_i T$ for $1 \leq i \leq t$ yields a new variable T at level j with coefficient $\pi^j m$. We call this process a *contraction of variables to a variable at level j.*

Contractions are useful for finding nontrivial zeros of additive forms due to the following version of Hensel's Lemma. This is the standard version of Hensel's lemma for finite extensions of $\mathbb{Q}_p$, written in the language of contractions and specialized to our particular fields and polynomial F. For a more general discussion of Hensel's Lemma, see [7].

Lemma 4. *Suppose that F is an additive form as in* (2) *with coefficients in $\mathcal{O}_K$. Let x_i be a variable at level h. Suppose that x_i can be used in a contraction of variables (or in one of a series of contractions) which produces a new variable at level at least $h+5$. Then F has a nontrivial zero in $\mathcal{O}_K$.*

In the proof of the theorem, our goal will be to show that if F is a normalized form, then there exists a variable which can be moved up by at least 5 levels via contractions. This will generally, although not always, be a variable which originates at level 0. To this end, we make the following definitions. A variable at level 0 will be called *primary*, as will any variable which is formed by a contraction which uses a primary variable. All other variables will be referred to as *secondary* variables. With this definition, one consequence of Hensel's Lemma is that if we can use contractions to create a primary variable at level at least 5, then F has a nontrivial zero in $\mathcal{O}_K$.

3. A lower bound

In this section, we show that $\Gamma^*(6, K)$ is at least as large as the bounds given in Section 1 for each of the fields K which we are considering. We do this by giving explicit forms which have no nontrivial solutions in K. It turns out that we can use essentially the same form for each of the four fields for which $\Gamma^*(6, K) = 9$. Let K be one of these fields, and let π be the uniformizer for K defined in Section 1. Then the form

$$x_1^6 + x_2^6 + x_3^6 + (1+\pi)x_4^6 + \pi^2(1+\pi)(x_5^6 + x_6^6 + x_7^6) + \pi^4 x_8^6$$

has no nontrivial zeros in K. To prove this, we can check that in order to make this form congruent to 0 modulo π^6, every variable must be divisible by π. Therefore, all zeros in K must have every variable divisible by π. However, since the form is homogeneous, we should be able to cancel factors of π from all the variables of any nontrivial zero until at least one variable is no longer divisible by π, a contradiction. Since there exist diagonal forms in 8 variables with no nontrivial zeros, we must have $\Gamma^*(6, K) \geq 9$ for each of these fields.

For the other two fields $K = \mathbb{Q}_2(\sqrt{-1})$ and $K = \mathbb{Q}_2(\sqrt{-5})$, we can write down an explicit form in 6 variables with no nontrivial zeros. For these two fields, the form

$$\left(x_1^6 + (1+\pi)x_2^6\right) + \pi^2\left(x_3^6 + (1+\pi)x_4^6\right) + \pi^4\left(x_5^6 + (1+\pi)x_6^6\right)$$

has no nontrivial zeros in K, which can be shown in the same way as above. This shows that $\Gamma^*(6, K) \geq 7$ for these two fields.

4. Preliminary Lemmata

In this section, we state a number of preliminary lemmata that we will use in the proof of the theorem. Half of these lemmata describe when we may make certain types of contractions, while the other half give some simple situations where we can show that F must have a nontrivial zero. Note that the hypotheses and proofs of these lemmata involve the π-coefficient of a variable, which was defined in Section 2.

Our first lemma in this section is the key to our proof of Theorem 1, and will be used many times. Since Lemma 5 will be used so often, we will typically not explicitly cite it when it is used. If we simply state that a contraction can be made without mentioning any lemma as justification, then it will be Lemma 5 that allows us to make the contraction.

Lemma 5. *Suppose that x and y are variables at level k. If x and y have different π-coefficients, then they can be contracted to a variable T at level $k+1$. Moreover, we can arrange so that T has whichever π-coefficient we like. If x and y have the same π-coefficient, then they can be contracted to a variable T at level $k+2$. Also, in this case they can be contracted to a variable T at level at least $k+3$.*

We note that in the case where x and y have the same π-coefficient, we cannot control the π-coefficient of T. Moreover, if we contract to level at least $k+3$, then we cannot control the exact level of T.

Proof. Without loss of generality, we may assume that both x and y are at level 0. Let α be an element of $\mathcal{O}_K$ such that $\alpha^6 \equiv 1+\pi^2 \pmod{\pi^3}$. (Note from Table 1 that such an element exists.) Suppose first that x and y have different π-coefficients. Looking modulo π^3, we may assume that their terms in F look like

$$(1+c_1\pi+c_2\pi^2)x^6+(1+d_1\pi+d_2\pi^2)y^6,$$

where $c_1, c_2, d_1, d_2 \in \{0,1\}$ and $c_1 \neq d_1$. If we set $x=y=T$, then the coefficient of T^6 is

$$\pi(c_1+d_1)+\pi^2(1+c_2+d_2)=\pi(1+\pi[1+c_2+d_2]),$$

where we have used the fact that $2 \equiv \pi^2 \pmod{\pi^3}$. If we instead set $x=T$ and $y=\alpha T$, then the coefficient of T^6 is

$$\pi(c_1+d_1)+\pi^2(c_2+d_2)=\pi(1+\pi[c_2+d_2]).$$

Whichever choice we make, this coefficient is divisible by π, but not by π^2, and so T is a variable at level 1. Also, the numbers $c_2 + d_2$ and $1 + c_2 + d_2$ are different modulo 2, and so these two possible contractions produce variables with different π-coefficients.

Now suppose instead that x and y have the same π-coefficient, so that $c_1 = d_1$. If we set $x = y = T$, then the coefficient (modulo π^3) of T^6 will be

$$\pi(2c_1) + \pi^2(1 + c_2 + d_2) \equiv \pi^2(1 + c_2 + d_2).$$

If we instead set $x = T$ and $y = \alpha T$, then the coefficient of T^6 will be

$$\pi(2c_1) + \pi^2(c_2 + d_2) \equiv \pi^2(c_2 + d_2).$$

One of c_2+d_2 and $1+c_2+d_2$ will be congruent to $1 \pmod 2$, and so that change of variables produces a variable T at level 2. The other possible change of variables yields coefficient of T^6 which is zero modulo π^3, and so T will be a variable at level at least 3. $\square$

The following lemma is a trivial corollary of Lemma 5, but is convenient to state on its own.

Lemma 6. *Suppose that x and y are variables at level k with different π-coefficients. Moreover, suppose that there is at least one variable at each of level $k+1, \ldots, k+t$. Then we can use x and y in contractions which create a variable at level at least $k + t + 1$. Further, we may choose freely between any of the following options.*

- *We may create the new variable at level exactly $k + t + 1$, with whichever π-coefficient we like.*
- *We may create the new variable at level exactly $k + t + 2$, although we can no longer control its π-coefficient.*
- *We may create the new variable at level at least $k + t + 3$, although we can control neither the exact level nor the π-coefficient of this variable.*

Proof. Since x and y have different π-coefficients, Lemma 5 allows us to contract them to a variable at level $k+1$ whose π-coefficient is different from the π-coefficient of a variable which is already at this level. Then we may similarly contract our new variable and a variable at level $k+1$ to form a variable at level $k+2$, whose π-coefficient is different from the π-coefficient of a variable already at that level. We continue until we form a variable at level $k+t$. When we make this variable, we can choose it to have whichever π-coefficient we desire. If we wish to ultimately create a variable at level $k+t+1$, then we make the contracted variable at level $k+t$ have a different π-coefficient than a variable already there. If we wish to ultimately create a variable at a higher level, then we make the contracted variable at level $k+t$ have the same π-coefficient as a variable already there. Then a final appeal to Lemma 5 completes the proof.

□

Although it is not explicitly stated in the lemma, the proof makes the following fact clear. If z is another variable at any of levels $k+1,\ldots,k+t$, then we may arrange for z to also be used in the contraction.

The next two lemmata are more technical and are only needed once, when we deal with the situation in which $(m_0,m_1,m_2,m_3,m_4,m_5)=(3,0,3,0,3,0)$.

Lemma 7. *Suppose that F contains at least 3 variables at level k which all have the same π-coefficient. Moreover, suppose that if two of these variables are contracted to a new variable T at level $k+2$, then T will have the same π-coefficient no matter which variables were selected to make the contraction. If K is one of the fields $\mathbb{Q}_2(\sqrt{2}),\mathbb{Q}_2(\sqrt{-2}),\mathbb{Q}_2(\sqrt{10})$, and $\mathbb{Q}_2(\sqrt{-10})$, then the π-coefficient of T will be the same as the π-coefficients of the original variables. If K is either $\mathbb{Q}_2(\sqrt{-1})$ or $\mathbb{Q}_2(\sqrt{-5})$, then the π-coefficient of T will be the opposite of the π-coefficients of the original variables.*

Proof. Without loss of generality, we may assume that $k=0$, and also that there are exactly 3 variables at level 0, which we label x_1,x_2,x_3. Since we are interested in the π-coefficient of a variable at level 2, we need to consider all of our coefficients modulo π^4. So assume that these variables appear in the form F as

$$(1+c\pi+d_1\pi^2+e_1\pi^3)x_1^6+(1+c\pi+d_2\pi^2+e_2\pi^3)x_2^6+(1+c\pi+d_3\pi^2+e_3\pi^3)x_3^6, \quad (5)$$

where we have $c, d_1, e_1, d_2, e_2, d_3, e_3 \in \{0,1\}$.

First, we prove the lemma for the fields $K = \mathbb{Q}_2(\sqrt{2}), \mathbb{Q}_2(\sqrt{-2}), \mathbb{Q}_2(\sqrt{10})$, and $\mathbb{Q}_2(\sqrt{-10})$. Note that for each of these fields, we have $2 \equiv \pi^2 \pmod{\pi^4}$, and also that the only nonzero sixth powers modulo π^4 are 1 and $1+\pi^2+\pi^3$. We now look at the results of contracting x_1 and x_2 to a new variable T. Let α be an element of $\mathcal{O}_K$ such that $\alpha^6 \equiv 1+\pi^2+\pi^3 \pmod{\pi^4}$. If we set $x_1 = x_2 = T$, then the coefficient of T, taken modulo π^4, will be

$$(1+d_1+d_2)\pi^2 + (c+e_1+e_2)\pi^3.$$

On the other hand, if we set $x_1 = T, x_2 = \alpha T$, then the coefficient of T, taken modulo π^4, will be

$$(d_1+d_2)\pi^2 + (1+e_1+e_2)\pi^3.$$

The results when we contract the other pairs of variables are similar.

We now split the proof into two cases, depending on the value of c. Suppose first that $c = 1$. If we have the same π-coefficient whenever we contract two variables to level 2, then we must have

$$1+e_1+e_2 \equiv 1+e_1+e_3 \equiv 1+e_2+e_3 \pmod{2}.$$

This implies that $e_1 = e_2 = e_3$, and so the π-coefficient of T will be 1, as desired.

Now suppose instead that $c = 0$. Suppose first that $d_1 = d_2 = d_3$. Then since only one π-coefficient is possible when we contract to a variable T at level 2, we must have

$$e_1+e_2 \equiv e_1+e_3 \equiv e_2+e_3 \pmod{2}.$$

This again implies that $e_1 = e_2 = e_3$, and so the π-coefficient of T will be 0, as desired. On the other hand, if d_1, d_2, d_3 are not all equal, then we may assume without loss of generality that $d_1 = d_2 \neq d_3$. Since only one π-coefficient is possible for the variable T at level 2, we must have

$$e_1 + e_2 \equiv 1 + e_1 + e_3 \equiv 1 + e_2 + e_3 \pmod{2}.$$

This yields $e_1 \equiv e_2 \equiv 1 + e_3 \pmod{2}$, which implies that the π-coefficient of T is 0. This completes the proof of the lemma for these four fields.

Now we prove the lemma for the fields $\mathbb{Q}_2(\sqrt{-1})$ and $\mathbb{Q}_2(\sqrt{-5})$. The proof here is a little bit different because now we have $2 \equiv \pi^2 + \pi^3 \pmod{\pi^4}$. However, it's still the case that the only sixth powers modulo π^4 are 1 and $1 + \pi^2 + \pi^3$. As before, we look at all the possibilities for contracting the variables x_1 and x_2. If we set $x_1 = x_2 = T$, then the reduced coefficient of T will be

$$(1 + d_1 + d_2)\pi^2 + (1 + c + e_1 + e_2)\pi^3.$$

If we set $x_1 = T, x_2 = \alpha T$, then the reduced coefficient of T will be

$$(d_1 + d_2)\pi^2 + (e_1 + e_2)\pi^3.$$

The results when we contract other pairs of variables are similar.

Again, the situation changes depending on the value of c. If $c = 1$, then the π-coefficient for T will always be $e_1 + e_2 \pmod{2}$. In this case, if we obtain the same π-coefficient no matter which pair of variables we contract, then we must have

$$e_1 + e_2 \equiv e_1 + e_3 \equiv e_2 + e_3 \pmod{2}.$$

As above, this implies that $e_1 \equiv e_2 \equiv e_3 \pmod{2}$, and so the π-coefficient of T must be 0. This is the opposite π-coefficient from the variables at level 0.

Finally, suppose that $c = 0$. Among the values of d_1, d_2, and d_3, at least two of them must be the same. Suppose without loss of generality that $d_1 = d_2$. If it happens that we also have $d_3 = d_1$, and only one π-coefficient is possible when we contract to level 2, then we must have

$$1 + e_1 + e_2 \equiv 1 + e_1 + e_3 \equiv 1 + e_2 + e_3 \pmod{2}.$$

This implies that $e_1 = e_2 = e_3$, and hence that the only possible π-coefficient for T is 1. On the other hand, if $d_3 \neq d_1$ and only one π-coefficient is possible for T, then we must have

$$1 + e_1 + e_2 \equiv e_1 + e_3 \equiv e_2 + e_3 \pmod{2}.$$

In this case, we find that $e_1 = e_2$ and $e_1 \neq e_3$, and again the only possible π-coefficient for T is 1, which is the opposite π-coefficient from the π-coefficients of the variables at level 0. This completes the proof of the lemma.

□

Lemma 8. *Suppose that F contains at least 3 variables at level k which all have the same π-coefficient. Suppose further that the field K is either $\mathbb{Q}_2(\sqrt{-1})$ or $\mathbb{Q}_2(\sqrt{-5})$. Then there are two variables at level k which can be contracted to a variable at level at least $k+4$.*

Proof. Without loss of generality, we may assume that $k = 0$. Note that for these two fields, we have $2 \equiv \pi^2 + \pi^3 \pmod{\pi^4}$. Also, the only sixth powers modulo π^4 are 1 and $1 + \pi^2 + \pi^3$. Let α be an element of $\mathcal{O}_K$ such that $\alpha^6 \equiv 1 + \pi^2 + \pi^3 \pmod{\pi^4}$. As in the proof of Lemma 7, assume that the variables at level 0 are x_1, x_2, and x_3, and that these variables appear in F as in (5) when their coefficients are reduced modulo π^4. We now look at all the possibilities for contracting the variables x_1 and x_2. If we set $x_1 = x_2 = T$, then the coefficient of T, reduced modulo π^4, will be

$$(1 + d_1 + d_2)\pi^2 + (1 + c + e_1 + e_2)\pi^3.$$

If we set $x_1 = T, x_2 = \alpha T$, then the reduced coefficient of T will be

$$(d_1 + d_2)\pi^2 + (e_1 + e_2)\pi^3.$$

Again, we get a similar result when we contract other pairs of variables.

Suppose first that $d_1 = d_2 = d_3$. Among our variables, there must be two for which the e-values are equal. Without loss of generality, suppose that $e_1 = e_2$. Then setting $x_1 = T, x_2 = \alpha T$ yields a new variable T whose coefficient is 0 when reduced modulo π^4. That is, the variable T is at level at least 4.

Suppose instead that the d-values are not all equal. Then two of them must be equal, so suppose that $d_1 = d_2 \neq d_3$. If $e_1 = e_2$, then we can contract these variables to level at least 4 as above, and we are done. So assume instead that $e_1 \neq e_2$. Now, if we make a contraction by setting $x_1 = x_3 = T$, then the reduced coefficient of T would be $(1 + c + e_1 + e_3)\pi^3$. On the other hand, if we set $x_2 = x_3 = S$, then the reduced coefficient of S would be $(1 + c + e_2 + e_3)\pi^3$. Since $e_1 \neq e_2$, one of the terms in parentheses must be 0 (mod 2), and the corresponding contraction yields a variable at level at least 4.

$\square$

Our final four lemmata in this section describe simple situations under which we can guarantee that the form F in (2) has a nontrivial zero in K.

Lemma 9. *Suppose that F contains two variables at level k with the same π-coefficient, a variable at level $k+3$, and a variable at level $k+4$. Then F has a nontrivial zero in K.*

Proof. By Lemma 5, we can contract the variables at level k to a variable at level at least $k + 3$. If this new variable is at level at least $k + 5$, then we stop. If it is at level $k+4$, then it and the variable already at level $k+4$ contract to a variable at level at least $k + 5$. If the new variable is at level $k + 3$, then we use it and the variable already there to make a contraction. If the resulting variable is at level $k + 4$, then we contract it with the variable already there to yield a variable at level at least $k + 5$. Hence we are always able to use variables from level k in contractions which yield a variable at level at least $k + 5$. The conclusion now follows by Hensel's Lemma.

□

Lemma 10. *Suppose that F contains four variables at level k, and that they can be split into two pairs such that the variables in each pair have the same π-coefficient. Suppose also that F has one additional variable whose level can be any of $k+2$, $k+3$, or $k+4$. Then F has a nontrivial zero in K.*

PROOF. Without loss of generality, we may assume that $k=0$. By Hensel's Lemma, our goal is to create a primary variable at level at least 5. Call the additional variable x, and suppose first that x is at level 4. In this case, we contract each pair of variables at level 0 to a primary variable at level at least 3. Call these variables S and T. If either of S and T is at level 4, then it can be contracted with x to form a primary variable at level at least 5. If both S and T are at level 3, then we can contract them. If this contraction only produces a new variable at level 4, then this variable and x contract to a primary variable at level at least 5.

Suppose instead that x is at level 3. Again, we can use the variables at level 0 to form two primary variables S and T at level at least 3. If both of these variables are at level 4, then we contract them, and are done. If S is at level 3 and T is at level 4, then we contract S and x. If necessary, we then contract the resulting primary variable with T, producing a primary variable at level at least 5. If S and T are both at level 3, then we can find two variables at level 3 with the same π-coefficient, and at least one of them is primary. These two variables then contract to level at least 5.

Finally, assume that x is at level 2. Again, we contract each pair of variables at level 0, but this time we contract to two primary variables S and T at level 2. We now have 3 variables at level 2. As above, there must be a pair with the same π-coefficient, and at least one of them must be primary. This pair of variables can be contracted to a primary variable at level at least 5.

□

Lemma 11. *Suppose that we have $m_0 \geq 2$, $m_1 \geq 2$, and $m_2 \geq 1$. Then F has a nontrivial zero in K.*

Proof. Suppose first that there are two variables at level 0 which have different π-coefficients. Then Lemma 6 immediately allows us to use contractions to create a primary variable at level at least 5. So suppose instead that the variables at level 0 all have the same π-coefficient. Contract two of these variables to form a primary variable at level 2. If it has the same π-coefficient as a secondary variable at level 2, then we can contract them to form a primary variable at level at least 5, and we are done. So suppose that this primary variable has a different π-coefficient from any secondary variable at level 2. Now, consider the variables at level 1. If there are two with different π-coefficients, then we can contract them to a secondary variable at level 2 which has the same π-coefficient as the primary variable there, and we can then contract these variables to make a primary variable at level at least 5. In the final case, if the variables at level 1 all have the same π-coefficient, then we can contract two of them to level exactly 3. Next, then we can use Lemma 6 on the variables at levels 2 and 3 to form a primary variable at level at least 5. Since we are always able to construct a primary variable at level at least 5, we are finished by Hensel's Lemma.

$\square$

Lemma 12. *Suppose that F is π-normalized and that m_3 and m_4 are both nonzero. Then F has a nontrivial zero in K.*

Proof. By normalization, we know that $m_0 \geq 2$. If there are two variables at level 0 with the same π-coefficient, then we are finished by Lemma 9. Otherwise, we must have $m_0 = 2$, and then the normalization properties (4) guarantee that $m_1 \geq 1$. Since the variables at level 0 have different π-coefficients, Lemma 6 allows us to construct a primary variable at level at least 4. If it is at level exactly 4, then we may contract it with a variable already there to produce a primary variable at level at least 5. This completes the proof of the lemma.

$\square$

5. The proof of the theorem: Levels with lots of variables

We now begin the proof of the theorem. We need to show that any additive form F as in (2) must have a nontrivial zero. In this section, we show that if F has "many" variables at the same level, then the form must have a nontrivial zero. For the cases in this section, we do not need the full power of normalization. We assume throughout this section that every variable is at level at most 5, but we do not need the properties (4) about the sums of the m_i. For all of the cases in this section, we are able to treat all of the possible fields K at the same time.

Lemma 13. *Suppose that F has at least 7 variables at the same level. Then F has a nontrivial zero in K.*

Proof. Suppose that F has at least 7 variables at level k. Among these variables, there must be three pairs such that the variables in each pair have the same π-coefficient. Contract each of these pairs to form 3 variables at level $k+2$. Two of these new variables must have the same π-coefficient, and therefore may be contracted to form a new variable at level at least $k+5$. The conclusion now follows from Hensel's lemma.

□

Lemma 14. *If F has at least 6 variables at the same level, then F has a nontrivial zero in K.*

Proof. Suppose that the 6 variables are at level k, and let $x_1, \ldots, x_t$ be all the variables (if any) at levels less than k. Then making the change of variables

$$F' = \frac{1}{\pi^k} F(\pi x_1, \ldots \pi x_t, x_{t+1}, \ldots, x_9)$$

moves the 6 variables to level 0, while preserving the property that all variables are at level at most 5. Hence, we may assume at the beginning that the 6 variables are at level 0. As always, our goal is to use Hensel's lemma, usually by producing a primary variable at level at least 5. Suppose that among the variables at level 0, we may find three disjoint pairs such that the variables in each pair have the same

π-coefficients. Then we may proceed exactly as in Lemma 13 to show that F has a nontrivial zero. So we may assume instead that we may separate the 6 variables into 3 disjoint pairs such that the variables in two of the pairs have the same π-coefficient, and the variables in the other pair have different π-coefficients. We now divide the proof into cases according to which levels contain the remaining variables.

If $m_1 \geq 1$, then we contract the 3 pairs to form 2 primary variables at level 2 and one primary variable at level 1 whose π-coefficient is different than the π-coefficient of the variable already there. The two variables at level 1 can then be contracted to form a third primary variable at level 2. With three primary variables at level 2, we can proceed as in the proof of Lemma 13.

If $m_2 \geq 1$, we again begin by forming two primary variables at level 2. Then two of the three variables at level 2 must have the same π-coefficient, and at least one of these variables must be primary. Again, we can now finish as in the proof of Lemma 13.

If $m_3 \geq 1$, we again begin by forming two primary variables at level 2. If they have the same π-coefficient, then we can finish as above. Otherwise, we may contract them to a primary variable at level 3 which has the same π-coefficient as the variable already there. We can then contract these two variables to a primary variable at level 5, and we are done.

If $m_4 \geq 1$, then we begin by contracting two pairs of variables at level 0 with the same π-coefficient, but this time we form two primary variables at level at least 3. If either primary variable is at level 5, then we are finished. If either primary variable is at level 4, then we can contract it with the secondary variable already there to form a primary variable at level at least 5. If both of the primary variables are at level 3, then we can contract them to a primary variable either at level at least 5 (in which case we are done) or at level 4, in which case we are done after contracting it further with the secondary variable at level 4.

Finally, if none of the previous cases apply, then we must have $m_0 = 6$ and $m_5 = 3$. Suppose that $x_1, \ldots, x_6$ are the variables at level 0, and make the change of variables

$$F' = \pi^{-5} F(\pi x_1, \ldots, \pi x_6, x_7, x_8, x_9).$$

Then F' has 3 variables at level 0 and 6 variables at level 1. As in the previous case, we may contract two pairs of variables from level 1 to form variables at level at least 4. If either of these variables is at level at least 6, then we are finished by Hensel's Lemma. If both variables are at level 5, then they can be contracted to a variable at level at least 6, and again Hensel's Lemma finishes the proof. If neither of these cases occur, then we make our contractions from level 1 slightly differently. We know now that at we can contract one pair of variables to produce a new variable at level exactly 4. Then we contract the other pair to produce a variable at level 3. Now, since there are 3 variables at level 0, some two of them must have the same π-coefficient. Then the hypotheses of Lemma 9 are satisfied, showing that F' (and hence F) has a nontrivial zero.

$\square$

Lemma 15. *If F has 5 variables at the same level, then F has a nontrivial zero in K.*

Proof. As in the proof of Lemma 14, we may assume that the five variables are at level 0. We can find two pairs of these variables which have the same π-coefficient. Each of these pairs can be contracted to a primary variable at level at least 3. If both new variables are at level at least 4, or if both are at level 3 and have the same π-coefficient, then we can finish as in previous lemmata. So assume that either both new variables are at level 3 with different π-coefficients, or we have one new variable at level 3 and one at level 4. We now study the positions of the remaining variables.

Suppose that $m_4 \geq 1$. If the two new primary variables are at level 3 with different π-coefficients, then we can contract them to a primary variable at level

4. So in any case, we may construct a primary variable at level 4. This can be contracted with the secondary variable at level 4 to produce a primary variable at level at least 5, and we are done.

Suppose now that $m_3 \geq 1$. If two variables at level 3, at least one of which is primary, have the same π-coefficients, then we may contract them to produce a primary variable at level at least 5. Otherwise, we have one primary variable at level 3 and one at level 4, and the primary variable at level 3 has a different π-coefficient from a secondary variable there. Then Lemma 6 allows us to construct a primary variable at level at least 5, and we are done.

Next, suppose that $m_2 \geq 1$. In this case, instead of creating two primary variables at level at least 3, we create two primary variables at level 2. Combined with one secondary variable at level 2, we have a set of three variables. Two of them must have the same π-coefficient, and one of these must be primary. We can use these two variables to construct a primary variable at level at least 5, and we are finished.

Next, suppose that $m_1 \geq 2$ (*not* that $m_1 \geq 1$). Depending on the π-coefficients of the variables at level 1, they can be contracted to a secondary variable at level either 2 or 3. Either possibility puts us in one of the cases we previously considered, so the proof is complete in this situation.

If none of the previous cases apply, then we have $m_1 \leq 1$ and $m_5 \geq 3$. Suppose that $x_1, \ldots, x_t$ are the variables at level 0 and 1 (so that $5 \leq t \leq 6$), and make the change of variables

$$F' = \pi^{-5} F(\pi x_1, \ldots, \pi x_t, x_{t+1}, \ldots, x_9).$$

Then F' has at least 3 variables at level 0 and 5 variables at level 1. As before, within the 5 variables at level 1, we can find two pairs of variables having the same π-coefficients. Now we may finish by proceeding exactly as in the final case of the proof of Lemma 14.

□

6. The proof of the theorem: Small values of m_0

In this section, we complete the proof of Theorem 1 by showing that F must possess a nontrivial zero in the remaining cases. Here, we will make use of all the properties of normalization. Hence we will assume that all 9 variables are at level at most 5 and that all the inequalities in the system (4) hold. Note that we may assume that m_5 equals either 0 or 1, or else this system cannot be satisfied. By Lemma 12, we may assume that at least one of m_3 and m_4 equals 0, and the results of the previous section allow us to assume that $m_i \leq 4$ for all i. In particular, we may assume that $m_0 \leq 4$. We now give several lemmata to finish the proof of the theorem, basing our hypotheses on the number of variables at level 0.

Lemma 16. *Suppose that the form F is π-normalized and that $m_0 = 4$. Then F has a nontrivial zero.*

Proof. With $m_0 = 4$, the normalization inequalities (4) give us

$$m_1 + m_2 \geq 1, \qquad m_1 + m_2 + m_3 \geq 2, \qquad m_1 + m_2 + m_3 + m_4 \geq 4,$$

where at least one of m_3 and m_4 equals 0.

Suppose first that we may divide the variables at level 0 into two pairs such that within each pair, the variables have the same π-coefficient. If any of m_2, m_3, m_4 are nonzero, then Lemma 10 shows that F has a nontrivial zero. If $m_2 = m_3 = m_4 = 0$, then we have $m_1 \geq 4$. Then there must be two variables at level 1 with the same π-coefficient, and these can be contracted to a variable at level exactly 3. This again puts us in a situation in which Lemma 10 applies, and we see that F has a nontrivial zero. Therefore we may assume that at level

0, we have 3 variables with the same π-coefficient and 1 variable with the other π-coefficient. We split the rest of the proof into various cases.

Case A: $m_1 \geq 1$**.** Divide the variables at level 0 into two pairs such that the variables in one pair have the same π-coefficient and the variables in the other pair have different π-coefficients. Now, contract the pair with the same π-coefficient to a primary variable T at level 2. Then using Lemma 6, we may use contractions to create a primary variable at level at least 5. This completes the proof of this case.

Case B: $m_1 = 0$ **and** $m_3 \geq 1$**.** Since $m_1 = 0$, we must have $m_2 \geq 1$. We begin by contracting two variables from level 0 to a primary variable T at level 2. If T has the same π-coefficient as a secondary variable at level 2, then we contract these variables to a primary variable at level at least 5, and we are finished. On the other hand, if T has a different π-coefficient from some variable at level 2, then these variables can be contracted to a primary variable at level 3 which has the same π-coefficient as a secondary variable already there. Finally, we contract these variables to a primary variable at level at least 5, and we are again finished. This completes the proof of this case.

Case C: $m_1 = m_3 = 0$ **and** $m_2 = 4$**.** If there are variables at level 2 with different π-coefficients, then we may proceed as in the previous case to form a primary variable at level at least 5, and we are done. If all the variables at level 2 have the same π-coefficient, then since either m_4 or m_5 is nonzero, the hypotheses of Lemma 10 are satisfied, and so F has a nontrivial zero. This finishes the proof of this case.

Case D: $m_1 = m_3 = 0$**,** $m_2 \leq 3$**, and** $m_4 \geq 2$**.** First, we contract two variables at level 0 with the same π-coefficient (call these variables x_1, x_2) to form a variable T at level 2. Now assume the two variables remaining at level 0 are x_3, x_4 and note that they have different π-coefficients. Make the change of variables

$$F' = F(\pi x_3, \pi x_4, x_5, \ldots, x_9, T).$$

The effect of this change is to move x_3 and x_4 to level 6. Since we now have 3 variables at level 2, two of them must have the same π-coefficient, and we can contract them to a variable at level 4. This guarantees that we have at least 3 variables at level 4, and hence two of these must again have the same π-coefficient. Contract them to a variable S at level 6. Since the variables already at level 6 have different π-coefficients, one of them must have the same π-coefficient as S, and these may be contracted to a variable at level at least 9. Since some of the variables used in these contractions came from level 4 or lower, we have contracted at least one variable up by at least 5 levels. This finishes the proof of this case.

Case E: We have $(m_0, m_1, m_2, m_3, m_4, m_5) = (4, 0, 3, 0, 1, 1)$**.** After the previous cases, this is the only remaining possibility for the m_i. Suppose that the variables at level 0 are $x_1, \ldots, x_4$ and consider the form F' obtained by making the change of variables

$$F' = F(\pi x_1, \ldots, \pi x_4, x_5, \ldots, x_9).$$

This change moves all of the variables at level 0 to level 6. Now, since there are 3 variables at level 2, there must be two with the same π-coefficient. This puts us in the situation of Lemma 9, and so F' (and hence also F) has a nontrivial zero. This completes the proof of the lemma.

$\square$

Lemma 17. *Suppose that the form F is π-normalized and that $m_0 = 3$. Then F has a nontrivial zero, except possibly in the case where $m_0 = m_2 = m_4 = 3$ and $m_1 = m_3 = m_5 = 0$.*

Proof. First, note that if we have $m_1 \geq 2$ and $m_2 \geq 1$, then we are done by Lemma 11. Hence we may assume that either $m_1 \leq 1$ or $m_2 = 0$. Moreover, we may assume as before that either $m_3 = 0$ or $m_4 = 0$. As in the previous lemma, we split the proof into a number of cases, based on the values of the m_i.

Case A: $m_3 = 0$ **and** $m_4 = 0$**.** In this case, the normalization properties give $m_1 + m_2 \geq 5$. Since we may assume that no level contains more than 4 variables, we must have $m_1 = 1$, $m_2 = 4$, and $m_5 = 1$. If all of the variables at level 2 have the same π-coefficient, then Lemma 10 shows that F has a nontrivial zero, so we may assume that there are variables at level 2 with different π-coefficients. Among the variables at level 0, there must be two with the same π-coefficient. Contract these variables to a primary variable at level 2. This variable must have the same π-coefficient as one of the secondary variables at level 2, and so we may create a primary variable at level at least 5. This finishes the proof of this case. We assume throughout the rest of the proof that exactly one of m_3 and m_4 is nonzero.

Case B: $m_2 = 0$ and $m_3 = 0$. In this case, we must have $m_1 \geq 3$ and $m_4 \geq 1$. There must be two variables at level 1 with the same π-coefficient, and we may contract these variables to produce a secondary variable at level 3. Since there must be two variables at level 0 with the same π-coefficients, Lemma 9 shows that F has a nontrivial zero. This completes the proof of this case.

Case C: $m_2 = 0$ and $m_4 = 0$. In this case, we must have $m_1 \geq 2$, $m_3 \geq 1$, and $m_1 + m_3 \geq 5$. First, suppose that there are two variables at level 1 with the same π-coefficient, and contract them to a secondary variable S at level 3. If S has the same coefficient as a variable already at level 3, then we may contract them to a variable at level at least 6. Since a variable from level 1 was used in the contractions, we are finished by Hensel's Lemma. So we may assume that S has a different π-coefficient than any of the other variables at level 3. Now we contract two variables from level 0 to a primary variable T at level at least 3. If T is at level 3, then it has the same π-coefficient as a secondary variable already there, and we contract them to a primary variable at level at least 5. If T is at level 4, then we contract two variables at level 3 to a secondary variable at level 4, and then contract this with T to produce a primary variable at level at least 5.

Suppose instead that we cannot find two variables at level 1 with the same π-coefficient. In this case, we contract two variables at level 0 with the same π-coefficient to a variable at level 2. Then Lemma 6 allows us to create a primary variable at level at least 5. This finishes the proof of this case. Note that Cases

A, B, and C together allow us to assume that $m_2 > 0$, and therefore that $m_1 \leq 1$.

Case D: $m_1 \leq 1$ and $m_4 = 0$. Our assumptions in this case imply that $m_2 \geq 1$ and $m_3 \geq 1$. We can contract two variables at level 0 to a primary variable T at level 2. If T has the same π-coefficient as a secondary variable at level 2, then we can contract them to a primary variable at level at least 5. If T has a different π-coefficient than all of the secondary variables at level 2, then Lemma 6 allows us to construct a primary variable at level at least 5. This completes the proof of this case.

Case E: $m_1 \leq 1$, $m_2 = 4$, and $m_3 = 0$. Note that these conditions imply that $m_4 \geq 1$. We begin by contracting two of the variables from level 0 to a variable at level 2. After making this contraction, we have 5 variables at level 2 and at least one variable at level 4. Then the proof of Lemma 15 shows that F has a nontrivial zero. (Note that the proof of the relevant case of Lemma 15 does not require all 9 variables, and so it is not an issue that we only have 8 variables left after making the initial contraction.)

Case F: $m_1 \leq 1$, $m_3 = 0$, and there are variables at level 2 with different π-coefficients. In this case, we contract two variables from level 0 to a primary variable at level exactly 2. This has the same π-coefficient as a secondary variable already there, so we contract these variables to a primary variable at level at least 5.

Case G: $m_1 \leq 1$, $m_3 = 0$, and $m_5 = 1$. These conditions guarantee that $m_2 \geq 2$ and $m_4 \geq 1$. Moreover, after Case F, we may assume that the variables at level 2 all have the same π-coefficient. Contract two variables at level 2 to a new variable T at level 4. Then either Lemma 5 or Lemma 6 (depending on the π-coefficient of T) guarantees that T can be used in a contraction to a variable at lemma at least 7. Since a variable from level 2 has been contracted up at least 5 levels, Hensel's Lemma guarantees that F has a nontrivial zero. We may now assume for the rest of the proof that $m_5 = 0$. Along with all our other assumptions, and since we are done if $m_2 \geq 4$, this implies that we have $m_4 \geq 2$.

Case H: $m_1 = 1$ **and** $m_3 = 0$**.** Due to our previous work, we have $m_4 \geq 2$ and we know that the variables at level 2 all have the same π-coefficient. Suppose first that there are variables with different π-coefficients at level 4. Then we contract two variables at level 2 to a variable at level 4. This must have the same π-coefficient as a variable already there, and so we contract these variables to a variable at level at least 7. As before, we have now contracted a variable at level 2 up by at least 5 levels, and we are done.

If the variables at level 4 have the same π-coefficient, then we let $x_1, \ldots, x_t$ be the variables at levels 0-3 and make the change of variables

$$F' = \frac{1}{\pi^4} F(\pi x_1, \ldots, \pi x_t, x_{t+1}, \ldots, x_9).$$

The form F' may not be π-normalized, but has at least 2 variables with the same π-coefficient at level 0, and has variables at both level 3 and level 4. Then Lemma 9 shows that F' (and hence also F) has a nontrivial zero. This completes the proof of this case.

After the preceding cases, the only remaining possibility is that $m_0 = m_2 = m_4 = 3$ and $m_1 = m_3 = m_5 = 0$. This completes the proof of the lemma. $\square$

The next lemma completes the proof that F must have a nontrivial zero whenever $m_0 = 3$. We give this last situation its own lemma because we cannot treat all the possible fields K at the same time.

Lemma 18. *Suppose that F is π-normalized and that $m_0 = m_2 = m_4 = 3$ and $m_1 = m_3 = m_5 = 0$. Then F has a nontrivial zero.*

Proof. As shown in the proof of Lemma 17, we may assume that the variables at level 2 all have the same π-coefficient. Suppose that there exist variables at level 4 with different π-coefficients. Then we can contract two variables from level 2 to a variable T at level 4. Since T will have the same π-coefficient as a variable already there, these two variables can be contracted to a variable at level at least 7. Then we are finished by Hensel's Lemma. Therefore we may also assume that the variables at level 4 all have the same π-coefficient and that when we contract two variables from level 2 to level 4, the only possible π-coefficient for the contracted variable is the one opposite from the variables already at level 4.

We can make similar arguments, possibly after making the change of variables of the form

$$F' = \frac{1}{\pi^2} F(\pi x_1, \pi x_2, \pi x_3, x_4, \ldots, x_9),$$

to deal with the variables at level 0. These arguments show that we may assume that the variables at level 0 all have the same π-coefficient, and that if two of these variables are contracted to a variable at level 2, then the contracted variable must have the opposite π-coefficient from the variables already at this level. Similarly, if two variables at level 4 are contracted to a variable at level 6, then the contracted variable must have the opposite π-coefficient from the variables at level 0.

We now split into cases depending on the field K. Suppose first that K is one of the fields $\mathbb{Q}_2(\sqrt{2}), \mathbb{Q}_2(\sqrt{-2}), \mathbb{Q}_2(\sqrt{10})$, or $\mathbb{Q}_2(\sqrt{-10})$. Let c be the π-coefficient of the variables at level 0. Since only one π-coefficient is possible when two of these variables are contracted to a variable at level 2, Lemma 7 shows that this variable must have a π-coefficient of c. Since this must be the opposite π-coefficient from the variables already at level 2, the variables at level 2 must have a π-coefficient of $1-c$. Similarly, when two variables at level 2 are contracted to a variable at level 4, Lemma 7 shows that the contracted variable must have a π-coefficient of $1-c$, and hence the variables at level 4 must have a π-coefficient of c. Finally, if two variables at level 4 are contracted to a new variable at level 6, this new variable must have a π-coefficient of c. But this contradicts our assumption that this variable should have the opposite π-coefficient from the variables at level 0. This proves the lemma for these fields.

Now suppose instead that K is either $\mathbb{Q}_2(\sqrt{-1})$ or $\mathbb{Q}_2(\sqrt{-5})$. Since all the variables at level 0 have the same π-coefficient, Lemma 8 says that we can find two of these variables which contract to a primary variable T at level at least 4. If T is at level exactly 4, then we may contract it with a secondary variable from this level to form a primary variable at level at least 5. This completes the proof of the lemma.

$\square$

Lemma 19. *Suppose that the form F is π-normalized and that $m_0 = 2$. Then F has a nontrivial zero.*

Proof. We begin this final lemma with a few observations. By the results of Section 5, we may assume that no level contains more than 4 variables. That is, we may assume that $m_i \leq 4$ for all i. Also, by Lemma 11, we may assume that either $m_1 \leq 1$ or $m_2 = 0$. Finally, by Lemma 12, we may assume that at most one of m_3 and m_4 is nonzero. We now split the proof into a number of cases.

Case A: $m_4 \geq 3$: In this case, our assumptions about the values of the m_i, combined with the restrictions in the system (4), force us to have $m_4 = 3$ and $m_5 = 0$. Suppose that the variables at level 4 are x_7, x_8, and x_9, and consider the additive form

$$F' = \frac{1}{\pi^4} F(\pi x_1, \ldots, \pi x_6, x_7, x_8, x_9).$$

This form has coefficients in $\mathcal{O}_K$, is π-normalized (in our sense), and has 3 variables at level 0. Therefore F' (and hence F) has a nontrivial zero by Lemma 17. In the rest of the proof of the lemma, we may now assume that $m_4 \leq 2$.

Case B: $m_2 = 0$ and $m_3 = 0$. In this case, our assumptions about the m_i force us to have $(m_0, m_1, m_2, m_3, m_4, m_5) = (2, 4, 0, 0, 2, 1)$. Since $m_1 = 4$, there must be two variables at level 1 which have the same π-coefficient. Then the hypotheses of Lemma 9 are satisfied, and so F must have a nontrivial zero.

Case C: $m_2 = 0$, and $m_4 = 0$. In this case, we must have $m_1 \geq 3$ and $m_3 \geq 2$. Suppose first that the variables at level 0 have different π-coefficients. Contract these variables to a primary variable at level 1 in such a way that there are two pairs of variables at level 1 with the same π-coefficients. Contract each of these pairs to form two variables T_1 and T_2 at level exactly 3. Now, consider the variables T_1, T_2, and any other variable at level 3. Two of these variables must have the same π-coefficient, and at least one of these must have been made using a variable from level 1. We can contract these two variables to form a variable at level at least 6. This final variable was made using a variable from level 1, and so Hensel's Lemma guarantees that F has a nontrivial zero.

If the variables at level 0 have the same π-coefficient, then we begin by considering the variables at level 1 instead. There must be two of these which have the same π-coefficient. Contract these variables to form a variable at level exactly 3. If this has the same π-coefficient as a variable already at level 3, then we contract these variables to a variable at level at least 6, and we are done. Otherwise, we now have two secondary variables at level 3 with different π-coefficients. Contract the variables at level 0 to form a primary variable at level at least 3. If this primary variable is at level exactly 3, then it has the same π-coefficient as one of the secondary variables there, and we can contract to a primary variable at level at least 5. If the variables from level 0 contract to a primary variable at level 4, then we can use Lemma 6 to contract the variables at levels 3 and 4 to a primary variable at level 5. This completes the proof of this case.

Case D: $m_2 \neq 0$**.** In this case, normalization and our assumptions about the m_i force us to have $m_1 = 1$ and $m_2 \geq 2$. If we actually have $m_2 \geq 3$, then suppose that x_1, x_2, x_3 are the variables at levels 0 and 1, and consider the form

$$F' = \frac{1}{\pi^2} F(\pi x_1, \pi x_2, \pi x_3, x_4, \ldots, x_9).$$

The form F' is π-normalized (in our sense), and has at least 3 variables at level 0. So F' (and hence F) has a nontrivial zero by either Lemma 16 or Lemma 17.

Finally, suppose that $m_2 = 2$. Then normalization guarantees that $m_3 \geq 1$. In fact, since $m_3 \neq 0$, we must have $m_4 = 0$, and hence the normalization inequalities (4) actually give $m_3 \geq 3$. Again, let x_1, x_2, x_3 be the variables at levels 0 and 1, and consider the form

$$F' = \frac{1}{\pi^2} F(\pi x_1, \pi x_2, \pi x_3, x_4, \ldots, x_9).$$

The form F' is normalized and has 2 variables at level 0, no variables at level 2, and 2 variables at level 4. If F' has any variables at level 3, then it has a nontrivial zero by Lemma 12. Otherwise, F' has no variables at level 3, and therefore has a nontrivial zero by Case B of this lemma. Since F' has a nontrivial zero, F must have one as well. This completes the proof of the lemma, and also the proof of the theorem. $\square$

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